

# Supplementary Material of Exact Recoverability of Robust PCA via Outlier Pursuit with Tight Recovery Bounds

Hongyang Zhang<sup>†</sup>, Zhouchen Lin<sup>†\*</sup>, Chao Zhang<sup>†\*</sup>, Edward Y. Chang<sup>‡</sup>

<sup>†</sup>Key Lab. of Machine Perception (MOE), School of EECS, Peking University, China

<sup>‡</sup> HTC Research, Taiwan

{hy\_zh, zlin}@pku.edu.cn, chzhang@cis.pku.edu.cn, edward.chang@htc.com

Robust PCA via Outlier Pursuit:

$$\min_{L,S} \|L\|_* + \lambda \|S\|_{2,1}, \quad \text{s.t. } M = L + S. \quad (1)$$

$\mu$ -incoherence condition on matrix  $L = U\Sigma V^T$ :

$$\max_i \|V^T e_i\|_2 \leq \sqrt{\frac{\mu r}{n}}, \quad (\text{avoid column sparsity}) \quad (2a)$$

$$\max_i \|U^T e_i\|_2 \leq \sqrt{\frac{\mu r}{m}}, \quad (\text{avoid row sparsity}) \quad (2b)$$

$$\|UV^T\|_\infty \leq \sqrt{\frac{\mu r}{mn}}. \quad (2c)$$

Ambiguity condition on matrix  $S$ :

$$\|\mathcal{B}(S)\| \leq \sqrt{\log n}/4. \quad (3)$$

Main Results:

**Theorem 1** (Exact Recovery of Outlier Pursuit). *Suppose  $m = \Theta(n)$ ,  $\text{Range}(L_0) = \text{Range}(\mathcal{P}_{\mathcal{I}_0^\perp} L_0)$ , and  $[S_0]_{:j} \notin \text{Range}(L_0)$  for  $\forall j \in \mathcal{I}_0$ . Then any solution  $(L_0 + H, S_0 - H)$  to Outlier Pursuit (1) with  $\lambda = 1/\sqrt{\log n}$  exactly recovers the column space of  $L_0$  and the column support of  $S_0$  with a probability at least  $1 - cn^{-10}$ , if the column support  $\mathcal{I}_0$  of  $S_0$  is uniformly distributed among all sets of cardinality  $s$  and*

$$\text{rank}(L_0) \leq \rho_r \frac{n(2)}{\mu \log n} \quad \text{and} \quad s \leq \rho_s n, \quad (4)$$

where  $c$ ,  $\rho_r$ , and  $\rho_s$  are constants,  $L_0 + \mathcal{P}_{\mathcal{I}_0} \mathcal{P}_{\mathcal{U}_0} H$  satisfies  $\mu$ -incoherence condition (2a), and  $S_0 - \mathcal{P}_{\mathcal{I}_0} \mathcal{P}_{\mathcal{U}_0} H$  satisfies ambiguity condition (3).

## Architecture of Proofs

This section is devoted to proving Theorem 1. Without loss of generality, we assume  $m = n$ . The following theorem shows that Outlier Pursuit succeeds for easy recovery problem.

**Theorem 2** (Elimination Theorem). *Suppose any solution  $(L^*, S^*)$  to Outlier Pursuit (1) with input  $M = L^* + S^*$  exactly recovers the column space of  $L_0$  and the column*

\*Corresponding authors.

Table 1: Summary of main notations used in the supplementary material.

Notations	Meanings
$m, n$	Size of the data matrix $M$ .
$n_{(1)}, n_{(2)}$	$n_{(1)} = \max\{m, n\}$ , $n_{(2)} = \min\{m, n\}$ .
$\Theta(n)$	Grows in the same order of $n$ .
$O(n)$	Grows equal to or less than the order of $n$ .
$e_i$	Vector whose $i$ th entry is 1 and others are 0s.
$M_{:j}$	The $j$ th column of matrix $M$ .
$M_{ij}$	The entry at the $i$ th row and $j$ th column of $M$ .
$\ \cdot\ _2$	$\ell_2$ norm for vector, $\ v\ _2 = \sqrt{\sum_i v_i^2}$ .
$\ \cdot\ _*$	Nuclear norm, the sum of singular values.
$\ \cdot\ _0$	$\ell_0$ norm, number of nonzero entries.
$\ \cdot\ _{2,0}$	$\ell_{2,0}$ norm, number of nonzero columns.
$\ \cdot\ _1$	$\ell_1$ norm, $\ M\ _1 = \sum_{i,j}  M_{ij} $ .
$\ \cdot\ _{2,1}$	$\ell_{2,1}$ norm, $\ M\ _{2,1} = \sqrt{\sum_j \ M_{:j}\ _2^2}$ .
$\ \cdot\ _{2,\infty}$	$\ell_{2,\infty}$ norm, $\ M\ _{2,\infty} = \max_j \ M_{:j}\ _2$ .
$\ \cdot\ _F$	Frobenious norm, $\ M\ _F = \sqrt{\sum_{i,j} M_{ij}^2}$ .
$\ \cdot\ _\infty$	Infinity norm, $\ M\ _\infty = \max_{i,j}  M_{ij} $ .
$\ \mathcal{P}\ $	(Matrix) operator norm.
$L^*, S^*$	Optimal solutions to Outlier Pursuit.
$L_0, S_0$	Ground Truth.
$\hat{U}, \hat{V}$	Left and right singular vectors of $\hat{L}$ .
$\mathcal{U}_0, \hat{\mathcal{U}}, \mathcal{U}^*$	Column space of $L_0, \hat{L}, L^*$ .
$\mathcal{V}_0, \hat{\mathcal{V}}, \mathcal{V}^*$	Row space of $L_0, \hat{L}, L^*$ .
$\hat{\mathcal{T}}$	Space $\hat{\mathcal{T}} = \{\hat{U}X^T + Y\hat{V}^T, \forall X, Y \in \mathbb{R}^{n \times r}\}$ .
$\mathcal{X}^\perp$	Orthogonal complement of the space $\mathcal{X}$ .
$\mathcal{P}_{\hat{\mathcal{U}}}, \mathcal{P}_{\hat{\mathcal{V}}}$	$\mathcal{P}_{\hat{\mathcal{U}}}M = \hat{U}\hat{U}^T M$ , $\mathcal{P}_{\hat{\mathcal{V}}}M = M\hat{V}\hat{V}^T$ .
$\mathcal{P}_{\hat{\mathcal{T}}^\perp}$	$\mathcal{P}_{\hat{\mathcal{T}}^\perp}M = \mathcal{P}_{\hat{\mathcal{U}}^\perp} \mathcal{P}_{\hat{\mathcal{V}}^\perp}M$ .
$\mathcal{I}_0, \hat{\mathcal{I}}, \mathcal{I}^*$	Index of outliers of $S_0, \hat{S}, S^*$ .
$ \mathcal{I}_0 $	Outliers number of $S_0$ .
$X \in \mathcal{I}$	The column support of $X$ is a subset of $\mathcal{I}$ .
$\mathcal{B}(\hat{S})$	Operator normalizing non-zero columns of $\hat{S}$ , $\mathcal{B}(\hat{S}) = \{H: \mathcal{P}_{\hat{\mathcal{T}}^\perp}(H) = 0; H_{:j} = \frac{\hat{S}_{:j}}{\ \hat{S}_{:j}\ _2}, j \in \hat{\mathcal{I}}\}$ .
$\sim \text{Ber}(p)$	Obeys Bernoulli distribution with parameter $p$ .
$\mathcal{N}(a, b^2)$	Gaussian distribution (mean $a$ and variance $b^2$ ).

support of  $S_0$ , i.e.,  $\text{Range}(L^*) = \text{Range}(L_0)$  and  $\{j : S^*_{:j} \notin \text{Range}(L^*)\} = \mathcal{I}_0$ . Then any solution  $(L^*, S^*)$  to (1) with input  $M' = L^* + \mathcal{P}_{\mathcal{I}} S^*$  succeeds as well, where  $\mathcal{I} \subseteq \mathcal{I}^* = \mathcal{I}_0$ .

*Proof.* Let  $(L'^*, S'^*)$  be the solution of (1) with input matrix  $M'$  and  $(L^*, S^*)$  be the solution of (1) with input matrix  $M$ . Then we have

$$\|L'^*\|_* + \lambda \|S'^*\|_{2,1} \leq \|L^*\|_* + \lambda \|\mathcal{P}_{\mathcal{I}} S^*\|_{2,1}.$$

Therefore

$$\begin{aligned} & \|L'^*\|_* + \lambda \|S'^* + \mathcal{P}_{\mathcal{I}^\perp \cap \mathcal{I}_0} S^*\|_{2,1} \\ & \leq \|L'^*\|_* + \lambda \|S'^*\|_{2,1} + \lambda \|\mathcal{P}_{\mathcal{I}^\perp \cap \mathcal{I}_0} S^*\|_{2,1} \\ & \leq \|L^*\|_* + \lambda \|\mathcal{P}_{\mathcal{I}} S^*\|_{2,1} + \lambda \|\mathcal{P}_{\mathcal{I}^\perp \cap \mathcal{I}_0} S^*\|_{2,1} \\ & = \|L^*\|_* + \lambda \|S^*\|_{2,1}. \end{aligned}$$

Note that

$$L'^* + S'^* + \mathcal{P}_{\mathcal{I}^\perp \cap \mathcal{I}_0} S^* = M' + \mathcal{P}_{\mathcal{I}^\perp \cap \mathcal{I}_0} S^* = M.$$

Thus  $(L'^*, S'^* + \mathcal{P}_{\mathcal{I}^\perp \cap \mathcal{I}_0} S^*)$  is optimal to problem with input  $M$  and by assumption we have

$$\text{Range}(L'^*) = \text{Range}(L^*) = \text{Range}(L_0),$$

$$\{j : [S'^* + \mathcal{P}_{\mathcal{I}^\perp \cap \mathcal{I}_0} S^*]_{:j} \notin \text{Range}(L_0)\} = \text{Supp}(S_0).$$

The second equation implies  $\mathcal{I} \subseteq \{j : S'^*_{:j} \notin \text{Range}(L_0)\}$ . Suppose  $\mathcal{I} \neq \{j : S'^*_{:j} \notin \text{Range}(L_0)\}$ . Then there exists an index  $k$  such that  $S'^*_{:k} \notin \text{Range}(L_0)$  and  $k \notin \mathcal{I}$ , i.e.,  $M'_{:k} = L^*_{:k} \in \text{Range}(L_0)$ . Note that  $L'^*_{:j} \in \text{Range}(L_0)$ . Thus  $S'^*_{:k} \in \text{Range}(L_0)$  and we have a contradiction. Thus  $\mathcal{I} = \{j : S'^*_{:j} \notin \text{Range}(L_0)\} = \{j : S'^*_{:j} \notin \text{Range}(L^*)\}$  and the algorithm succeeds.  $\square$

Theorem 2 shows that the success of the algorithm is monotone on  $|\mathcal{I}_0|$ . Thus by standard arguments in (Candès et al. 2011), (Candès, Romberg, and Tao 2006), and (Candès and Tao 2010), any guarantee proved for the Bernoulli distribution equivalently holds for the uniform distribution. For completeness, we give the details in the appendix. In the following, we will assume  $\mathcal{I}_0 \sim \text{Ber}(p)$ .

There are two main steps in our following proofs: 1. find dual conditions under which Outlier Pursuit succeeds; 2. construct dual certificates which satisfy the dual conditions.

## Dual Conditions

We first give dual conditions under which Outlier Pursuit succeeds.

**Lemma 1** (Dual Conditions for Exact Column Space). *Let  $(L^*, S^*) = (L_0 + H, S_0 - H)$  be any solution to Outlier Pursuit (1),  $\hat{L} = L_0 + \mathcal{P}_{\mathcal{I}_0} \mathcal{P}_{\mathcal{U}_0} H$  and  $\hat{S} = S_0 - \mathcal{P}_{\mathcal{I}_0} \mathcal{P}_{\mathcal{U}_0} H$ , where  $\text{Range}(L_0) = \text{Range}(\mathcal{P}_{\mathcal{I}_0} L_0)$  and  $[S_0]_{:j} \notin \text{Range}(L_0)$  for  $\forall j \in \mathcal{I}_0$ . Assume that  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\| < 1$ ,  $\lambda > 4\sqrt{\mu r/n}$ , and  $\hat{L}$  obeys incoherence (2a). Then  $L^*$  has the same column space as that of  $L_0$  and  $S^*$  has the same column indices as those of  $S_0$  (thus  $\mathcal{I}_0 = \{j : S^*_{:j} \notin \text{Range}(L^*)\}$ ), provided that there exists a pair  $(W, F)$  obeying*

$$W = \lambda(\mathcal{B}(\hat{S}) + F), \quad (5)$$

with  $\mathcal{P}_{\hat{\mathcal{V}}} W = 0$ ,  $\|W\| \leq 1/2$ ,  $\mathcal{P}_{\hat{\mathcal{I}}} F = 0$  and  $\|F\|_{2,\infty} \leq 1/2$ .

*Proof.* We first recall that the subgradients of nuclear norm and  $\ell_{2,1}$  norm are as follows:

$$\partial_{\hat{\mathcal{I}}} \|\hat{L}\|_* = \{\hat{U} \hat{V}^T + \hat{Q} : \hat{Q} \in \hat{\mathcal{T}}^\perp, \|\hat{Q}\| \leq 1\},$$

$$\partial_{\hat{\mathcal{S}}} \|\hat{S}\|_{2,1} = \{\mathcal{B}(\hat{S}) + \hat{E} : \hat{E} \in \hat{\mathcal{I}}^\perp, \|\hat{E}\|_{2,\infty} \leq 1\}.$$

Let  $H_1 = \mathcal{P}_{\mathcal{I}_0} \mathcal{P}_{\mathcal{U}_0} H$  and  $H_2 = \mathcal{P}_{\mathcal{I}_0^\perp} \mathcal{P}_{\mathcal{U}_0} H + \mathcal{P}_{\mathcal{I}_0^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H + \mathcal{P}_{\mathcal{I}_0} \mathcal{P}_{\mathcal{U}_0^\perp} H$ , and note that  $\hat{U} = \mathcal{U}_0$  and  $\hat{\mathcal{I}} = \mathcal{I}_0$ . By the definition of the subgradient, the inequality follows

$$\begin{aligned} & \|L_0 + H\|_* + \lambda \|S_0 - H\|_{2,1} \\ & \geq \|\hat{L}\|_* + \lambda \|\hat{S}\|_{2,1} + \langle \hat{U} \hat{V}^T + \hat{Q}, H_2 \rangle - \lambda \langle \mathcal{B}(\hat{S}) + \hat{E}, H_2 \rangle \\ & = \|\hat{L}\|_* + \lambda \|\hat{S}\|_{2,1} + \langle \hat{U} \hat{V}^T, \mathcal{P}_{\mathcal{I}_0^\perp} H \rangle + \langle \hat{Q}, \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle - \\ & \quad \lambda \langle \mathcal{B}(\hat{S}), \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle - \lambda \langle \hat{E}, \mathcal{P}_{\mathcal{I}_0^\perp} H \rangle \\ & \geq \|\hat{L}\|_* + \lambda \|\hat{S}\|_{2,1} - \sqrt{\frac{\mu r}{n}} \|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1} + \langle \hat{Q}, \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle - \\ & \quad \lambda \langle \mathcal{B}(\hat{S}), \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle - \lambda \langle \hat{E}, \mathcal{P}_{\mathcal{I}_0^\perp} H \rangle. \end{aligned}$$

Now adopt  $\hat{Q}$  such that  $\langle \hat{Q}, \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle = \|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_*$  and  $\langle \hat{E}, \mathcal{P}_{\mathcal{I}_0^\perp} H \rangle = -\|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1}$ <sup>1</sup>. We have

$$\begin{aligned} & \|L_0 + H\|_* + \lambda \|S_0 - H\|_{2,1} \\ & \geq \|\hat{L}\|_* + \lambda \|\hat{S}\|_{2,1} - \sqrt{\frac{\mu r}{n}} \|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1} + \|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_* - \\ & \quad \lambda \langle \mathcal{B}(\hat{S}), \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle + \lambda \|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1} \\ & \geq \|\hat{L}\|_* + \lambda \|\hat{S}\|_{2,1} + \left(\frac{\lambda}{4} - \sqrt{\frac{\mu r}{n}}\right) \|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1} + \\ & \quad \|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_* - \lambda \langle \mathcal{B}(\hat{S}), \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle + \frac{3\lambda}{4} \|\mathcal{P}_{\hat{\mathcal{I}}^\perp} H\|_{2,1} \\ & \geq \|\hat{L}\|_* + \lambda \|\hat{S}\|_{2,1} + \left(\frac{\lambda}{4} - \sqrt{\frac{\mu r}{n}}\right) \|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1} + \\ & \quad \|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_* - \lambda \langle \mathcal{B}(\hat{S}), \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle + \frac{3\lambda}{4} \|\mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_{2,1}. \end{aligned}$$

Notice that

$$\begin{aligned} & |-\lambda \mathcal{B}(\hat{S}), \mathcal{P}_{\mathcal{U}_0^\perp} H| = |\langle \lambda F - W, \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle| \\ & \leq |\langle W, \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle| + \lambda |\langle F, \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle| \\ & \leq \frac{1}{2} \|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_* + \frac{\lambda}{2} \|\mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_{2,1}. \end{aligned}$$

Hence

$$\begin{aligned} & \|L_0 + H\|_* + \lambda \|S_0 - H\|_{2,1} \\ & \geq \|\hat{L}\|_* + \lambda \|\hat{S}\|_{2,1} + \left(\frac{\lambda}{4} - \sqrt{\frac{\mu r}{n}}\right) \|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1} + \\ & \quad \frac{1}{2} \|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_* + \frac{\lambda}{4} \|\mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_{2,1}. \end{aligned}$$

<sup>1</sup>By the duality between the nuclear norm and the operator norm, there exists a  $Q$  such that  $\langle Q, \mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H \rangle = \|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_*$  and  $\|Q\| \leq 1$ . Thus we take  $\hat{Q} = \mathcal{P}_{\mathcal{U}_0^\perp} \mathcal{P}_{\hat{\mathcal{V}}^\perp} Q \in \hat{\mathcal{T}}^\perp$ . It holds similarly for  $\hat{E}$ .

Since  $(L^*, S^*) = (L_0 + H, S_0 - H)$  is optimal, above inequality shows  $\|\mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_* = \|\mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\mathcal{U}_0^\perp} H\|_{2,1} = 0$ , i.e.,  $\mathcal{P}_{\mathcal{U}_0^\perp} H \in \hat{\mathcal{I}} \cap \hat{\mathcal{V}}$ . Also notice that  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\| < 1$  implies  $\hat{\mathcal{I}} \cap \hat{\mathcal{V}} = \{\mathbf{0}\}$ . We conclude  $\mathcal{P}_{\mathcal{U}_0^\perp} H = \mathbf{0}$ . Furthermore,  $\|\mathcal{P}_{\mathcal{I}_0^\perp} H\|_{2,1} = 0$  implies  $H \in \mathcal{I}_0$ . Thus  $H \in \mathcal{U}_0 \cap \mathcal{I}_0$ , i.e.,  $\mathcal{U}^* \subseteq \mathcal{U}_0$  and  $\mathcal{I}^* \subseteq \mathcal{I}_0$ .

We now prove  $\mathcal{U}^* = \mathcal{U}_0$ . According to the assumption  $\text{Range}(L_0) = \text{Range}(P_{\mathcal{I}_0^\perp} L_0)$  and  $H \in \mathcal{U}_0 \cap \mathcal{I}_0$ ,  $\text{Range}(L^*) = \text{Range}(L_0 + H) = \text{Range}(L_0)$ , i.e.,  $\mathcal{U}^* = \mathcal{U}_0$ . We then prove  $\mathcal{I}^* = \mathcal{I}_0$ . Assume that  $\mathcal{I}^* \neq \mathcal{I}_0$ , i.e., there exists a  $j \in \mathcal{I}_0$  such that  $S_{\cdot:j}^* = \mathbf{0}$ . Note that  $[S_0]_{\cdot:j} \notin \text{Range}(L_0)$ . Thus  $M_{\cdot:j} = [L_0]_{\cdot:j} + [S_0]_{\cdot:j} = L_{\cdot:j}^* \notin \mathcal{U}_0$ , which contradicts  $\mathcal{U}^* \subseteq \mathcal{U}_0$ . So  $\mathcal{I}^* = \mathcal{I}_0$ .  $\square$

**Remark 1.** *There are two important modifications in our conditions compared with those of (Xu, Caramanis, and Sanghavi 2012): 1. The space  $\hat{\mathcal{T}}$  (see Table 1) is not involved in our conclusion. Instead, we restrict  $W$  in the complementary space of  $\hat{\mathcal{V}}$ . The subsequent proofs benefit from such a modification. 2. Our conditions slightly simplify the constraint  $\hat{U} \hat{V}^T + W = \lambda(\mathcal{B}(\hat{S}) + F)$  in (Xu, Caramanis, and Sanghavi 2012), where  $\hat{U}$  is another dual certificate which needs to be constructed. Moreover, our modification enables us to build the dual certificate  $W$  by least squares and greatly facilitates our proofs.*

By Lemma 1, to prove the exact recovery of Outlier Pur-suit, it is sufficient to find a suitable  $W$  such that

$$\begin{cases} W \in \hat{\mathcal{V}}^\perp, \\ \|W\| \leq 1/2, \\ \mathcal{P}_{\hat{\mathcal{I}}} W = \lambda \mathcal{B}(\hat{S}), \\ \|\mathcal{P}_{\hat{\mathcal{I}}^\perp} W\|_{2,\infty} \leq \lambda/2. \end{cases} \quad (6)$$

As shown in the following proofs, our dual certificate  $W$  can be constructed by least squares.

### Certification by Least Squares

The remainder of the proofs is to construct  $W$  which satisfies dual conditions (6). Note that  $\hat{\mathcal{I}} = \mathcal{I}_0 \sim \text{Ber}(p)$ . To construct  $W$ , we consider the method of least squares, which is

$$W = \lambda \mathcal{P}_{\hat{\mathcal{V}}^\perp} \sum_{k \geq 0} (\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}})^k \mathcal{B}(\hat{S}). \quad (7)$$

Note that we have assumed  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\| < 1$ . Thus  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}}\| = \|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}} (\mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}})\| = \|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\|^2 < 1$  and equation (7) is well defined. We want to highlight the advantage of our construction over that of (Candès et al. 2011). In our construction, we use a smaller space  $\hat{\mathcal{V}} \subset \hat{\mathcal{T}}$  instead of  $\hat{\mathcal{T}}$  in (Candès et al. 2011). Such a utilization significantly facilitates our proofs. To see this, notice that  $\hat{\mathcal{I}} \cap \hat{\mathcal{T}} \neq \mathbf{0}$ . Thus  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{T}}}\| = 1$  and the Neumann series  $\sum_{k \geq 0} (\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{T}}} \mathcal{P}_{\hat{\mathcal{I}}})^k$  in the construction of (Candès et al. 2011) diverges. However, this issue does not exist for our construction. This benefits from our modification in Lemma 1. Moreover, our following theorem gives a good bound on  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\|$ , whose

proof takes into account that the elements in the same column of  $\hat{S}$  are not independent. The complete proof can be found in Appendices.

**Theorem 3.** *For any  $\mathcal{I} \sim \text{Ber}(a)$ , with an overwhelming probability*

$$\|\mathcal{P}_{\hat{\mathcal{V}}} - a^{-1} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\mathcal{I}} \mathcal{P}_{\hat{\mathcal{V}}}\| < \varepsilon, \quad (8)$$

*provided that  $a \geq C_0 \varepsilon^{-2} (\mu r \log n) / n$  for some numerical constant  $C_0 > 0$  and other assumptions in Theorem 1 hold.*

By Theorem 3, our bounds in Theorem 1 guarantee that  $a$  is always larger than a constant when  $\rho_r$  is selected small enough.

We now bound  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\|$ . Note  $\hat{\mathcal{I}}^\perp \sim \text{Ber}(1-p)$ . Then by Theorem 3, we have  $\|\mathcal{P}_{\hat{\mathcal{V}}} - (1-p)^{-1} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\hat{\mathcal{V}}}\| < \varepsilon$ , or equivalently  $(1-p)^{-1} \|\mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}} - p \mathcal{P}_{\hat{\mathcal{V}}}\| < \varepsilon$ . Therefore, by the triangle inequality

$$\begin{aligned} \|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\|^2 &= \|\mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\| \\ &\leq \|\mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}} - p \mathcal{P}_{\hat{\mathcal{V}}}\| + \|p \mathcal{P}_{\hat{\mathcal{V}}}\| \\ &\leq (1-p)\varepsilon + p. \end{aligned} \quad (9)$$

Thus we establish the following bound on  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\|$ .

**Corollary 1.** *Assume that  $\hat{\mathcal{I}} \sim \text{Ber}(p)$ . Then with an overwhelming probability  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\|^2 \leq (1-p)\varepsilon + p$ , provided that  $1-p \geq C_0 \varepsilon^{-2} (\mu r \log n) / n$  for some numerical constant  $C_0 > 0$ .*

Note that  $\mathcal{P}_{\hat{\mathcal{I}}} W = \lambda \mathcal{B}(\hat{S})$  and  $W \in \hat{\mathcal{V}}^\perp$ . So to prove the dual conditions (6), it is sufficient to show that

$$\begin{aligned} \text{(a)} \quad \|W\| &\leq 1/2, \\ \text{(b)} \quad \|\mathcal{P}_{\hat{\mathcal{I}}^\perp} W\|_{2,\infty} &\leq \lambda/2. \end{aligned} \quad (10)$$

### Proofs of Dual Conditions

Since we have constructed the dual certificates  $W$ , the remainder is to prove that the construction satisfies our dual conditions (10), as shown in the following lemma.

**Lemma 2.** *Assume that  $\hat{\mathcal{I}} \sim \text{Ber}(p)$ . Then under the other assumptions of Theorem 1,  $W$  given by (7) obeys the dual conditions (10).*

*Proof.* Let  $\mathcal{G} = \sum_{k \geq 0} (\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}})^k$ . Then

$$W = \lambda \mathcal{P}_{\hat{\mathcal{V}}^\perp} \sum_{k \geq 0} (\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}})^k \mathcal{B}(\hat{S}). \quad (11)$$

Now we check the two conditions in (10).

(a) By the assumption, we have  $\|\mathcal{B}(\hat{S})\| \leq \sqrt{\log n} / 4$ . Thus

$$\|W\| \leq \lambda \|\mathcal{P}_{\hat{\mathcal{V}}^\perp}\| \|\mathcal{G}\| \|\mathcal{B}(\hat{S})\| \leq 1/2. \quad (12)$$

(b) We have that  $W = \lambda \mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{G}(\mathcal{B}(\hat{S}))$ . Notice that  $\mathcal{G}(\mathcal{B}(\hat{S})) \in \hat{\mathcal{I}}$ . Thus

$$\begin{aligned} \mathcal{P}_{\hat{\mathcal{I}}^\perp} W &= \lambda \mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\hat{\mathcal{V}}^\perp} \mathcal{G}(\mathcal{B}(\hat{S})) \\ &= \lambda \mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{G}(\mathcal{B}(\hat{S})) - \lambda \mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{G}(\mathcal{B}(\hat{S})) \\ &= -\lambda \mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{G}(\mathcal{B}(\hat{S})) \\ &= -\lambda \mathcal{P}_{\hat{\mathcal{I}}^\perp} \mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}} \mathcal{G}(\mathcal{B}(\hat{S})). \end{aligned}$$

Denote by  $G = \sum_{k \geq 0} (I_{\hat{\mathcal{I}}} \hat{V} \hat{V}^T I_{\hat{\mathcal{I}}})^k$ , where  $I_{\hat{\mathcal{I}}}$  is a sub-sampling of columns of identity matrix at coordinates  $\hat{\mathcal{I}}$ . Then for any  $e_{j_0}$ , we have

$$\begin{aligned} \|(\mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}} \mathcal{G} \mathcal{B}(\hat{S})) e_{j_0}\|_2 &= \|\mathcal{B}(\hat{S}) G I_{\hat{\mathcal{I}}} \hat{V} \hat{V}^T e_{j_0}\|_2 \\ &\leq \|\mathcal{B}(\hat{S})\| \|G\| \|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\| \|\hat{V}^T e_{j_0}\|_2 \\ &\leq \frac{\sqrt{\log n}}{4} \frac{1}{1-\sigma} \sigma \sqrt{\frac{\mu r}{n}} \\ &\leq \frac{1}{2}, \end{aligned}$$

where the second inequality holds because  $\|\mathcal{P}_{\hat{\mathcal{I}}} \mathcal{P}_{\hat{\mathcal{V}}}\| \leq \sigma$ , which can be arbitrarily small constant by Corollary 1. Thus  $\|\mathcal{P}_{\hat{\mathcal{I}}^\perp} W\|_{2,\infty} \leq \lambda \|\mathcal{P}_{\hat{\mathcal{V}}} \mathcal{P}_{\hat{\mathcal{I}}} \mathcal{G} \mathcal{B}(\hat{S})\|_{2,\infty} \leq \lambda/2$ .  $\square$

Now we have proved that  $W$  satisfies the dual conditions (10). So our proofs are completed.

### Tightness of Bounds

The following theorem shows that our bounds in inequalities (4) are optimal.

**Theorem 4.** *The orders of the upper bounds given by inequalities (4) are tight.*

*Proof.* Since  $O(n)$  is the highest order for the possible number of corruptions, the order of our bound for the corruption cardinality  $s$  is tight.

We then demonstrate that our bound for  $\text{rank}(L_0)$  is tight. (McCoy and Tropp 2011) showed that the optimal solution  $L^*$  to model (1) satisfies

$$\text{rank}(L^*) \leq n\lambda^2 = n/\log n. \quad (13)$$

If the order of  $\text{rank}(L_0)$  is strictly higher than  $\Theta(n/\log n)$ , then according to (13) it is impossible for  $L^*$  to exactly recover the column space of  $L_0$  due to their different ranks. So  $\text{rank}(L_0)$  should be no larger than  $\Theta(n/\log n)$  and the order of our bound is tight.  $\square$

### Algorithm

In this section, we give the algorithm for Robust PCA (R-PCA) via Outlier Pursuit. To solve the model, we apply the alternating direction method (ADM) (Lin, Liu, and Su 2011), which is probably the most widely used method for solving nuclear norm minimization problems.

Given the Outlier Pursuit model

$$\min_{L,S} \|L\|_* + \lambda \|S\|_{2,1}, \quad \text{s.t. } M = L + S, \quad (14)$$

whose augmented Lagrangian formulation corresponds to

$$\begin{aligned} &\mathcal{L}(L, S, Y, \mu) \\ &= \|L\|_* + \lambda \|S\|_{2,1} + \langle M - L - S, Y \rangle + \frac{\mu}{2} \|M - L - S\|_F^2. \end{aligned} \quad (15)$$

ADM solves model (14) by updating one argument in (15) and fixing others in each step. For any matrix  $X$ , denote  $\mathcal{S}_\varepsilon$  and  $\mathcal{H}_\varepsilon$  the soft-thresholding operators on  $X$  such that

$$[\mathcal{S}_\varepsilon(X)]_{ij} = \begin{cases} X_{ij} - \varepsilon, & \text{if } X_{ij} > \varepsilon; \\ X_{ij} + \varepsilon, & \text{if } X_{ij} < -\varepsilon; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$[\mathcal{H}_\varepsilon(X)]_{:j} = \begin{cases} \frac{\|X_{:j}\|_2 - \varepsilon}{\|X_{:j}\|_2} X_{:j}, & \text{if } \|X_{:j}\|_2 > \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

The detailed procedures of the ADM are listed in the following algorithm:

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#### Algorithm 1 The ADM for R-PCA via Outlier Pursuit

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**Input:** Observation matrix  $M \in \mathbb{R}^{m \times n}$ ,  $\lambda = 1/\sqrt{\log n}$ .  
**Initialize:**  $Y_0 = \mathbf{0}$ ;  $L_0 = M$ ;  $S_0 = \mathbf{0}$ ;  $\mu_0 > 0$ ;  $k = 0$ .  
**1: while** not converged **do**  
**2:** //Line 3-4 solve  $L_{k+1} = \arg \min_L \mathcal{L}(L, S_k, Y_k, \mu_k)$ .  
**3:**  $(U, S, V) = \text{svd}(M - S_k + \mu_k^{-1} Y_k)$ ;  
**4:**  $L_{k+1} = U S \mu_k^{-1} (S) V^T$ .  
**5:** //Line 6 solves  $S_{k+1} = \arg \min_S \mathcal{L}(L_{k+1}, S, Y_k, \mu_k)$ .  
**6:**  $S_{k+1} = \mathcal{H}_{\lambda \mu_k^{-1}}[M - L_{k+1} + \mu_k^{-1} Y_k]$ .  
**7:**  $Y_{k+1} = Y_k + \mu_k (M - L_{k+1} - S_{k+1})$ .  
**8:** Update  $\mu_k$  to  $\mu_{k+1}$ .  
**9:**  $k \leftarrow k + 1$ .  
**10: end while**  
**Output:**  $(L^*, S^*)$ .

---

## Appendices

### Equivalence of Probabilistic Models

We show that the exact recovery result proved for the Bernoulli distribution holds for the uniform distribution as well. Let "success" be the event that the algorithm succeeds, i.e.,  $\text{Range}(L_0) = \text{Range}(L^*)$  and  $\{j : S_{:j}^* \notin \text{Range}(L^*)\} = \mathcal{I}_0$ . Notice the fact that

$$\mathbb{P}_{\text{Ber}(p)}(\text{Success} \mid |\mathcal{I}| = k) = \mathbb{P}_{\text{Unif}(k)}(\text{Success}),$$

and Theorem 2 implies that for  $k \geq t$ ,

$$\mathbb{P}_{\text{Unif}(k)}(\text{Success}) \leq \mathbb{P}_{\text{Unif}(t)}(\text{Success}).$$

Thus we have

$$\begin{aligned} \mathbb{P}_{\text{Ber}(p)}(\text{Success}) &= \sum_{k=0}^n \mathbb{P}_{\text{Ber}(p)}(\text{Success} \mid |\mathcal{I}| = k) \mathbb{P}_{\text{Ber}(p)}(|\mathcal{I}| = k) \\ &\leq \sum_{k=0}^{t-1} \mathbb{P}_{\text{Ber}(p)}(|\mathcal{I}| = k) + \sum_{k=t}^n \mathbb{P}_{\text{Ber}(p)}(\text{Success} \mid |\mathcal{I}| = k) \mathbb{P}_{\text{Ber}(p)}(|\mathcal{I}| = k) \\ &\leq \sum_{k=0}^{t-1} \mathbb{P}_{\text{Ber}(p)}(|\mathcal{I}| = k) + \sum_{k=t}^n \mathbb{P}_{\text{Unif}(k)}(\text{Success}) \mathbb{P}_{\text{Ber}(p)}(|\mathcal{I}| = k) \\ &\leq \mathbb{P}_{\text{Ber}(p)}(|\mathcal{I}| < t) + \mathbb{P}_{\text{Unif}(t)}(\text{Success}). \end{aligned}$$

Taking  $p = t/n + \varepsilon$  gives  $\mathbb{P}_{\text{Ber}(p)}(|\mathcal{I}| < t) \leq \exp(-\frac{\varepsilon^2 n}{2p})$ , we complete the proof.

### Proof of Theorem 3

We proceed to prove Theorem 3. The following lemma is critical.

**Lemma 3.** Assume  $\left\| \sum_{ij} y_{ij} \otimes y_{ij} \right\| \leq 1$  for  $y_{ij} \in \mathbb{R}^d$  and  $\delta_j$ s are i.i.d. Bernoulli variables with  $\mathbb{P}(\delta_j = 1) = a$ . Then

$$\mathbb{E} \left[ a^{-1} \left\| \sum_j (\delta_j - a) \sum_i y_{ij} \otimes y_{ij} \right\| \right] \leq \tilde{C} \sqrt{\frac{\log d}{a}} \max_{ij} \|y_{ij}\|,$$

provided that  $\tilde{C} \sqrt{\log d/a} \max_{ij} \|y_{ij}\| < 1$ .

*Proof.* Let

$$Y = \sum_j (\delta_j - a) \sum_i y_{ij} \otimes y_{ij},$$

and let  $Y' = \sum_j (\delta'_j - a) \sum_i y_{ij} \otimes y_{ij}$  be an independent copy of  $Y$ . Since  $\delta_j - \delta'_j$  is symmetric,  $Y - Y'$  has the same distribution as

$$Y_\varepsilon - Y'_\varepsilon \triangleq \sum_{ij} \varepsilon_{ij} (\delta_j - \delta'_j) y_{ij} \otimes y_{ij},$$

where  $\varepsilon_{ij}$ s are i.i.d. Rademacher variables and

$$Y_\varepsilon = \sum_{ij} \varepsilon_{ij} \delta_j y_{ij} \otimes y_{ij}.$$

Notice that  $\|\cdot\|$  is a convex function and  $\mathbb{E}_{\delta'} Y' = \mathbf{0}$ . Thus by Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}_\delta \|Y\| &= \mathbb{E}_\delta \|Y - \mathbb{E}_{\delta'} Y'\| \\ &= \mathbb{E}_\delta \|\mathbb{E}_{\delta'} (Y - Y')\| \\ &\leq \mathbb{E}_\delta \mathbb{E}_{\delta'} \|Y - Y'\| \\ &= \mathbb{E} \|Y_\varepsilon - Y'_\varepsilon\| \\ &\leq \mathbb{E} \|Y_\varepsilon\| + \mathbb{E} \|Y'_\varepsilon\| \\ &= 2\mathbb{E} \|Y_\varepsilon\| \\ &= 2\mathbb{E} \left\| \sum_{ij} \varepsilon_{ij} \delta_j y_{ij} \otimes y_{ij} \right\|. \end{aligned}$$

According to Rudelson's lemma in (Rudelson 1999), which states that

$$\begin{aligned} &\mathbb{E}_\varepsilon \left\| \sum_{ij} \varepsilon_{ij} \delta_j y_{ij} \otimes y_{ij} \right\| \\ &\leq C \sqrt{\log d} \max_{ij} \|y_{ij}\| \left\| \sum_{ij} \delta_j y_{ij} \otimes y_{ij} \right\|^{\frac{1}{2}}, \end{aligned}$$

we have

$$\begin{aligned} &\mathbb{E}_\delta \mathbb{E}_\varepsilon \left\| \sum_{ij} \varepsilon_{ij} \delta_j y_{ij} \otimes y_{ij} \right\| \\ &\leq C \sqrt{\log d} \max_{ij} \|y_{ij}\| \mathbb{E}_\delta \left\| \sum_{ij} \delta_j y_{ij} \otimes y_{ij} \right\|^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E} \|Y\| &\leq 2C \sqrt{\log d} \max_{ij} \|y_{ij}\| \mathbb{E}_\delta \left\| \sum_{ij} \delta_j y_{ij} \otimes y_{ij} \right\|^{\frac{1}{2}} \\ &\leq 2C \sqrt{\log d} \max_{ij} \|y_{ij}\| \sqrt{\mathbb{E} \left\| \sum_{ij} \delta_j y_{ij} \otimes y_{ij} \right\|} \\ &= 2C \sqrt{\log d} \max_{ij} \|y_{ij}\| \sqrt{\mathbb{E} \left\| \sum_j \delta_j \sum_i y_{ij} \otimes y_{ij} \right\|} \\ &= 2C \sqrt{\log d} \max_{ij} \|y_{ij}\| \sqrt{\mathbb{E} \left\| Y + a \sum_{ij} y_{ij} \otimes y_{ij} \right\|} \\ &\leq 2C \sqrt{\log d} \max_{ij} \|y_{ij}\| \sqrt{\mathbb{E} \|Y\| + a \left\| \sum_{ij} y_{ij} \otimes y_{ij} \right\|}. \end{aligned}$$

Thus we have

$$\begin{aligned} &a^{-1} \mathbb{E} \|Y\| \\ &\leq \frac{2C \sqrt{\log d}}{\sqrt{a}} \max_{ij} \|y_{ij}\| \sqrt{a^{-1} \mathbb{E} \|Y\| + \left\| \sum_{ij} y_{ij} \otimes y_{ij} \right\|} \\ &\leq \frac{2C \sqrt{\log d}}{\sqrt{a}} \max_{ij} \|y_{ij}\| \sqrt{a^{-1} \mathbb{E} \|Y\| + 1}. \end{aligned}$$

When  $2C \sqrt{\log d} \max_{ij} \|y_{ij}\| / \sqrt{a} < 1$ , then

$$\begin{aligned} a^{-1} \mathbb{E} \|Y\| &\leq 2 \frac{2C \sqrt{\log d}}{\sqrt{a}} \max_{ij} \|y_{ij}\| \\ &\triangleq \tilde{C} \sqrt{\frac{\log d}{a}} \max_{ij} \|y_{ij}\|, \end{aligned}$$

and the proof is completed.  $\square$

The following concentration inequality is also important to our proof of Theorem 3.

**Theorem 5** (Talagrand 1996). Assume that  $|f| \leq B$  and  $\mathbb{E} f(Y_i) = 0$  for every  $f$  in  $\mathcal{F}$ , where  $i = 1, \dots, n$  and  $\mathcal{F}$  is a countable family of functions such that if  $f \in \mathcal{F}$  then  $-f \in \mathcal{F}$ . Let  $Y_* = \sup_{f \in \mathcal{F}} \sum_{i=1}^n f(Y_i)$ . Then for any  $t \geq 0$ ,

$$\mathbb{P}(|Y_* - \mathbb{E} Y_*| > t) \leq 3 \exp \left( -\frac{t}{KB} \log \left( 1 + \frac{Bt}{\sigma^2 + B \mathbb{E} Y_*} \right) \right),$$

where  $\sigma^2 = \sup_{f \in \mathcal{F}} \sum_{i=1}^n \mathbb{E} f^2(Y_i)$ , and  $K$  is a constant.

Now we are ready to prove Theorem 3.

*Proof.* For any matrix  $X$ , we have

$$\mathcal{P}_{\hat{V}} X = \sum_{ij} \langle \mathcal{P}_{\hat{V}} X, e_i e_j^T \rangle e_i e_j^T.$$

Thus  $\mathcal{P}_{\mathcal{I}}\mathcal{P}_{\hat{V}}X = \sum_{ij} \delta_j \langle \mathcal{P}_{\hat{V}}X, e_i e_j^T \rangle e_i e_j^T$ , where where  $\delta_j$ s are i.i.d. Bernoulli variables with parameter  $a$ . Then

$$\begin{aligned} \mathcal{P}_{\hat{V}}\mathcal{P}_{\mathcal{I}}\mathcal{P}_{\hat{V}}X &= \sum_{ij} \delta_j \langle \mathcal{P}_{\hat{V}}X, e_i e_j^T \rangle \mathcal{P}_{\hat{V}}(e_i e_j^T) \\ &= \sum_{ij} \delta_j \langle X, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle \mathcal{P}_{\hat{V}}(e_i e_j^T). \end{aligned}$$

Namely,  $\mathcal{P}_{\hat{V}}\mathcal{P}_{\mathcal{I}}\mathcal{P}_{\hat{V}} = \sum_{ij} \delta_j \mathcal{P}_{\hat{V}}(e_i e_j^T) \otimes \mathcal{P}_{\hat{V}}(e_i e_j^T)$ . Now let

$$\begin{aligned} Z &= a^{-1} \left\| \mathcal{P}_{\hat{V}}\mathcal{P}_{\mathcal{I}}\mathcal{P}_{\hat{V}} - a\mathcal{P}_{\hat{V}} \right\| \\ &= a^{-1} \left\| \sum_{ij} (\delta_j - a) \mathcal{P}_{\hat{V}}(e_i e_j^T) \otimes \mathcal{P}_{\hat{V}}(e_i e_j^T) \right\|. \end{aligned}$$

We first prove the upper bound of  $\mathbb{E}Z$ . Adopt  $y_{ij} = \mathcal{P}_{\hat{V}}(e_i e_j^T)$  in Lemma 3. Since

$$\mathcal{P}_{\hat{V}} = \sum_{ij} \mathcal{P}_{\hat{V}}(e_i e_j^T) \otimes \mathcal{P}_{\hat{V}}(e_i e_j^T),$$

we have

$$\left\| \sum_{ij} \mathcal{P}_{\hat{V}}(e_i e_j^T) \otimes \mathcal{P}_{\hat{V}}(e_i e_j^T) \right\| = 1.$$

Thus by Lemma 3 and incoherence (2a),

$$\mathbb{E}Z \leq \tilde{C} \sqrt{\frac{\log n^2}{a}} \sqrt{\frac{\mu r}{n}} \triangleq C \sqrt{\frac{\mu r \log n}{na}}.$$

We then prove the upper bound of  $Z$  at an overwhelming probability. Let

$$\mathcal{D}_j = a^{-1}(\delta_j - a) \sum_i \mathcal{P}_{\hat{V}}(e_i e_j^T) \otimes \mathcal{P}_{\hat{V}}(e_i e_j^T),$$

and

$$\mathcal{D} = \sum_j \mathcal{D}_j = a^{-1}(\mathcal{P}_{\hat{V}}\mathcal{P}_{\mathcal{I}}\mathcal{P}_{\hat{V}} - a\mathcal{P}_{\hat{V}}).$$

Notice that the operator  $\mathcal{D}$  is self-adjoint. Denote the set  $g = \{\|X_1\|_F \leq 1, X_2 = \pm X_1\}$ . Then we have

$$\begin{aligned} Z &= \sup_g \langle X_1, \mathcal{D}(X_2) \rangle \\ &= \sup_g \sum_j \langle X_1, \mathcal{D}_j(X_2) \rangle \\ &= \sup_g \sum_j a^{-1}(\delta_j - a) \sum_i \langle X_1, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle \langle X_2, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle. \end{aligned}$$

Now let

$$\begin{aligned} f(\delta_j) &= \langle X_1, \mathcal{D}_j(X_2) \rangle \\ &= a^{-1}(\delta_j - a) \sum_i \langle X_1, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle \langle X_2, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle. \end{aligned}$$

To use Talagrand's concentration inequality on  $Z$ , we should bound  $|f(\delta_j)|$  and  $\mathbb{E}f^2(\delta_j)$ . Since by assumption,  $\hat{L} = L_0 +$

$\mathcal{P}_{\mathcal{I}_0}\mathcal{P}_{\mathcal{U}_0}H$  satisfies incoherence (2a) and

$$\begin{aligned} \|\mathcal{P}_{\hat{V}}X_1\|_{2,\infty}^2 &= \max_j \sum_i \langle X_1, e_i e_j^T \hat{V} \hat{V}^T \rangle^2 \\ &= \max_j \sum_i \langle e_i^T X_1, e_j^T \hat{V} \hat{V}^T \rangle^2 \\ &\leq \max_j \sum_i \|e_i^T X_1\|_2^2 \|e_j^T \hat{V} \hat{V}^T\|_2^2 \\ &= \max_j \|X_1\|_F^2 \|e_j^T \hat{V} \hat{V}^T\|_2^2 \\ &\leq \frac{\mu r}{n}, \end{aligned}$$

we have

$$\begin{aligned} |f(\delta_j)| &\leq a^{-1}|\delta_j - a| \sum_i |\langle X_1, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle| |\langle X_2, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle| \\ &= a^{-1}|\delta_j - a| \sum_i \langle X_1, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle^2 \\ &\leq a^{-1} \sum_i \langle \mathcal{P}_{\hat{V}}X_1, e_i e_j^T \rangle^2 \\ &\leq a^{-1} \|\mathcal{P}_{\hat{V}}X_1\|_{2,\infty}^2 \\ &\leq \frac{\mu r}{na}, \end{aligned}$$

where the first equality holds since  $X_2 = \pm X_1$ . Furthermore,

$$\begin{aligned} \mathbb{E}f^2(\delta_j) &= a^{-1}(1-a) \left( \sum_i \langle X_1, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle \langle X_2, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle \right)^2 \\ &\leq a^{-1}(1-a) \left( \sum_i |\langle X_1, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle| |\langle X_2, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle| \right)^2 \\ &= a^{-1}(1-a) \left( \sum_i \langle X_1, \mathcal{P}_{\hat{V}}(e_i e_j^T) \rangle^2 \right)^2 \\ &\leq a^{-1} \left( \sum_i \langle \mathcal{P}_{\hat{V}}X_1, e_i e_j^T \rangle^2 \right) \left( \sum_i \langle \mathcal{P}_{\hat{V}}X_1, e_i e_j^T \rangle^2 \right) \\ &\leq a^{-1} \|\mathcal{P}_{\hat{V}}X_1\|_{2,\infty}^2 \sum_i \langle \mathcal{P}_{\hat{V}}X_1, e_i e_j^T \rangle^2 \\ &\leq \frac{\mu r}{na} \sum_i \langle \mathcal{P}_{\hat{V}}X_1, e_i e_j^T \rangle^2, \end{aligned}$$

and

$$\begin{aligned} \sigma^2 &= \mathbb{E} \sum_j f^2(\delta_j) \leq \frac{\mu r}{na} \sum_{ij} \langle \mathcal{P}_{\hat{V}}X_1, e_i e_j^T \rangle^2 \\ &= \frac{\mu r}{na} \|\mathcal{P}_{\hat{V}}X_1\|_F^2 \\ &\leq \frac{\mu r}{na}. \end{aligned}$$

Since we have proved  $\mathbb{E}Z \leq 1$  in the first part of the proof, by Theorem 5,

$$\begin{aligned} \mathbb{P}(|Z - \mathbb{E}Z| > t) &\leq 3 \exp\left(-\frac{t}{KB} \log\left(1 + \frac{t}{2}\right)\right) \\ &\leq 3 \exp\left(-\frac{t \log 2}{KB} \min\left(1, \frac{t}{2}\right)\right), \end{aligned}$$

where the second inequality holds since  $\log(1 + u) \geq \log 2 \min(1, u)$  for any  $u \geq 0$ . Set

$$B = \frac{\mu r}{na} \quad \text{and} \quad t = \alpha \sqrt{\frac{\mu r \log n}{na}},$$

we have

$$\begin{aligned} & \mathbb{P} \left( |Z - \mathbb{E}Z| > \alpha \sqrt{\frac{\mu r \log n}{na}} \right) \\ & \leq 3 \exp \left( -\gamma_0 \min \left( 2\alpha \sqrt{\frac{na \log n}{\mu r}}, \alpha^2 \log n \right) \right) \\ & = 3 \exp(-\gamma_0 \alpha^2 \log n), \end{aligned}$$

where  $\gamma_0 = \log 2 / (2K)$  is a numerical constant. We now adopt  $\alpha = \sqrt{\beta / \gamma_0}$ . Thus

$$\mathbb{P} \left( |Z - \mathbb{E}Z| \leq \sqrt{\frac{\beta}{\gamma_0}} \sqrt{\frac{\mu r \log n}{na}} \right) \geq 1 - 3n^{-\beta}.$$

Note that we have proved  $\mathbb{E}Z \leq C \sqrt{\mu r \log n / na}$ . We have

$$\begin{aligned} \mathbb{P}(Z \leq \varepsilon) & \geq \mathbb{P} \left( Z \leq \sqrt{C_0} \sqrt{\frac{\mu r \log n}{na}} \right) \\ & = \mathbb{P} \left( Z \leq \left( C + \sqrt{\frac{\beta}{\gamma_0}} \right) \sqrt{\frac{\mu r \log n}{na}} \right) \\ & \geq \mathbb{P} \left( |Z - \mathbb{E}Z| \leq \sqrt{\frac{\beta}{\gamma_0}} \sqrt{\frac{\mu r \log n}{na}} \right) \\ & \geq 1 - 3n^{-\beta}, \end{aligned}$$

where  $C_0 \triangleq (C + \sqrt{\beta / \gamma_0})^2$  and the first inequality holds since  $a \geq C_0 \varepsilon^{-2} (\mu r \log n) / n$  by assumption. Thus the proof is completed.  $\square$

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